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*NILS BJÖRK*

**The Theory of the Indirectly Heated Thermistors.**

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# THEORY OF THE INDIRECTLY HEATED THERMISTOR

## A STUDY OF THERMISTOR CIRCUITS

4

BY

NILS BJÖRK

GÖTEBORG 1959
ELANDERS BOKTRYCKERI AKTIEBOLAG

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## Contents




**In the earlier Parts 1 and 2 of the series (Bibliography, Nos. 4 and 5) a theory of the directly heated thermistor has been given. In the present Part 4 this theory is extended to indirectly heated thermistors in order to make possible the handling of circuit problems where the dynamic properties of the indirectly heated thermistor are decisive.**

**In particular a small signal circuit for the indirectly heated thermistor is derived. The parameters of this circuit follow from an analytic expression of the resistance-power characteristic, which in its turn can be deduced from a few simple measurements.**

**The validity of the theory has been tested by experiments. The agreement is satisfactory.**

## List of Symbols

$t$ = time.
$\Theta$ = thermistor temperature.
$\Theta_0$ = ambient temperature.

$U$ = total thermistor voltage.
$I$ = total thermistor current.
$U_h$ = total heater voltage.
$I_h$ = total heater current.
$u$ = superimposed variable thermistor voltage.
$i$ = superimposed variable thermistor current.
$u_h$ = superimposed variable heater voltage.
$i_h$ = superimposed variable heater current.
$E_a$ = emf on heater side.
$E_b$ = emf on thermistor side.

$N = UI$ = power supplied to thermistor.
$N_h = U_h I_h$ = power supplied to heater.
$P$ = dissipated heat power.
$W$ = thermal energy of thermistor body.
$R = U/I$ = static resistance of thermistor.
$r = (dU/dI)_{N_h = \text{const.}}$ = dynamic resistance of thermistor.
$R_h = U_h/I_h$ = heater resistance.
$R_a$ = series resistance on heater side.
$R_b$ = series resistance on thermistor side.
$Z$ = thermistor impedance.

$\eta = -(dN/dN_h)_{R = \text{const.}}$ = heater efficiency.

$$F = -\left(\frac{dR/R}{dN/N}\right)_{N_h = \text{const.}} = -N/R \cdot dR/dP = \text{dynamic factor.}$$

$m = 1/R \cdot dR/dP$ = power coefficient of thermistor resistance.
$R_\infty$ = constant in the resistance equation (ohms).
$B$ = constant in the resistance equation (°K).
$C$ = power sensitivity (°K/mW).
$T$ = thermal time constant.

$$\tau = T\Big/\left(1 + F\,\frac{R_b - R_0}{R_b + R_0}\right) = \text{effective time constant of thermistor.}$$

Index 0 indicates value in operating point.

CHAPTER 1

# Introduction

An indirectly heated thermistor consists of a small bead of thermistor material ("the thermistor") with a large, negative temperature coefficient, surrounded by a heater with a very small temperature coefficient. The heater is electrically insulated from the thermistor by a thin layer of insulating cement. The device is enclosed in a small, evacuated or gas-filled glass bulb, see *Figs. 1* to *3*.

The resistance of the thermistor is determined by its temperature, which can be changed by supplying current to the heater, to the thermistor or to both. Of course, the thermistor temperature also depends on the ambient temperature. On account of the small dimensions of the thermistor, already a small amount of supplied

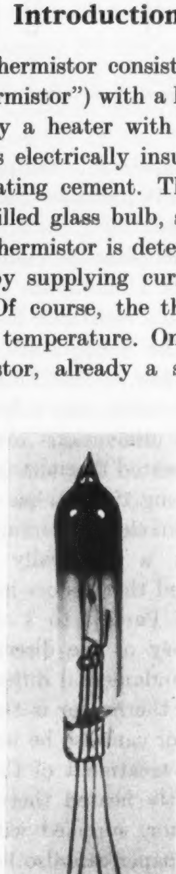


***Fig. 1.* Indirectly heated thermistor in evacuated glass bulb, Stantel type B 2552/60, approximately full size (manufacturer: Standard Telephones and Cables Ltd).**

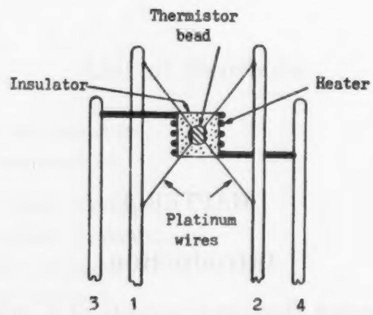



*Fig. 2.* Schematic picture of indirectly heated thermistor. The numbers refer to Fig. 3.

electric power gives a considerable temperature increase. Thus, a supplied power of only 3 mW increases the temperature of the thermistor in Fig. 1 with about 20° C and decreases its resistance to less than half the room temperature value.

This thermistor type has a maximum allowable dissipated power of 60 mW, corresponding to a resistance less than 1/1000 of the room temperature resistance. The temperature coefficient of the heater, on the other hand, is quite low, so that for normal operating temperatures the heater resistance varies only a few percent.

Great sensitivity, small dimensions and considerable sturdiness have made the indirectly heated thermistor widely employed in many fields [1, 2, 3, 6]. For a long time it has been used in applications where its properties as an electric circuit element are of decisive importance. Nevertheless, a practically useful theory of circuits containing indirectly heated thermistors has not yet been published.

In the earlier published Parts 1 to 3 of this study of thermistor circuits [4, 5, 7], the theory of the directly heated thermistor has been treated. The only fundamental difference between the directly and the indirectly heated thermistor is that the temperature of the indirectly heated thermistor can also be influenced through a heater. Therefore, the theoretical treatment of the two types has the same main features. The directly heated thermistor is equivalent to an indirectly heated thermistor, supplied with constant heater power. Hence, the results of this paper can also be immediately applied to a directly heated thermistor.

In *Chapter 2* the static properties of the indirectly heated thermistor are treated. It is shown that a few measurements are sufficient for

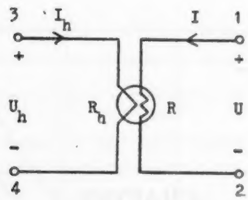



*Fig. 3.* Symbol for indirectly heated thermistor.

determining the data necessary for the solution of a static problem. In *Chapter 3* a derivation is given of the so-called thermistor equation, which governs the dynamic behaviour of the thermistor. An example shows the application of the thermistor equation to the solution of a dynamic problem. An equivalent small signal circuit of the indirectly heated thermistor is derived in *Chapter 4*. Some applications are given. The validity of the theoretical developments has been tested by an extended series of measurements, treated in *Chapter 5*.

The work described in this paper was made possible by grants from the Swedish Technical Research Council and from the Chalmers Research Fund. The author is indebted to Professor S. Ekelöf, Head of the Institute for Theoretical Electricity and Electrical Measurements, for having directed his interest towards the present problem and for valuable advice in the course of the work. He also wants to thank Tekn. lic. R. Davidson for stimulating discussions.

# CHAPTER 2

## Static Conditions

*a. Static characteristics*

**Throughout this paper the ambient temperature and other external conditions, which influence the thermistor temperature, are supposed to be constant.**

**The properties of the indirectly heated thermistor under static conditions are then completely defined by its static characteristics.**

**The static *U-I*-characteristic (*Fig. 4*) gives the relation between thermistor voltage *U* and thermistor current *I* with the heater current $I_h$ or the heater power $N_h$ as a parameter.**

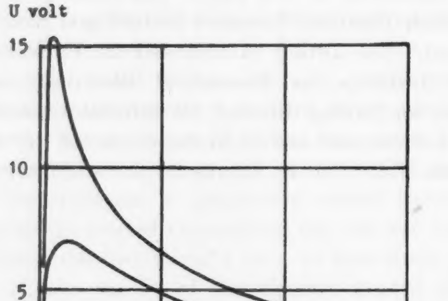



***Fig. 4.*** **Static *U-I*-characteristics with the heater power $N_h$ as a parameter. Thermistor Stantel type B 2552/60 at $\Theta_0 = 25°$ C.**

a) $N_h = 0$
b) $N_h = 10$ mW
c) $N_h = 20$ mW
d) $N_h = 30$ mW
e) $N_h = 40$ mW

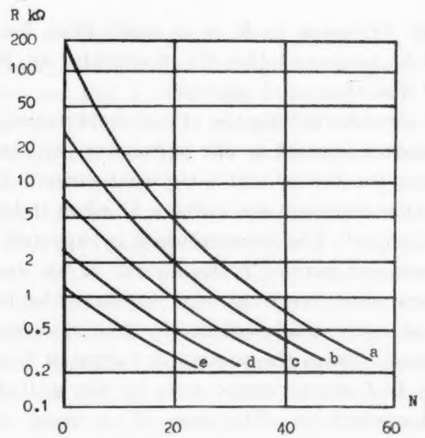



*Fig. 5.* **Static *R-N*-characteristics of the thermistor of Fig. 4.**

a) $N_h = 0$
b) $N_h = 10\,\mathrm{mW}$
c) $N_h = 20\,\mathrm{mW}$
d) $N_h = 30\,\mathrm{mW}$
e) $N_h = 40\,\mathrm{mW}$

The static *R-N*-characteristic (*Fig. 5*) gives the relation between the thermistor resistance $R$ and the supplied thermistor power $N$ with $I_h$ or $N_h$ as a parameter. Evidently, each of these characteristics can be calculated from the other by the relations

$$R = U/I; \quad N = UI \tag{1,2}$$

or, inversely

$$U = (NR)^{1/2}; \quad I = (N/R)^{1/2} \tag{3,4}$$

If we have an electric network containing a thermistor, we can use the static characteristics for calculating graphically in the usual way its currents and voltages.

If high accuracy is desired, we must sometimes pay regard to the fact that in spite of its small temperature coefficient the heater resistance $R_h$ increases noticeably with increasing thermistor temperature. It follows that the heater power $N_h = R_h\, I_h^2$ takes on different values in the different points of a static *U-I*-characteristic even if the heater current $I_h$ is kept constant. Hence the static characteristics with the parameter $N_h$ differ somewhat from those with the parameter

$I_h$. Generally, the variation in $R_h$ is so small that, for all practical purposes, it can be neglected (for the thermistor in Figs. 4 and 5 the variation is less than $\pm$ 1 percent).

The procedure for determining the static characteristics is obvious. If we want the heater current as the parameter, we send a constant current $I_h$ through the heater and a constant current $I$ through the thermistor. We then measure the voltage $U$ when it has reached its steady state end value. The measurement is repeated for a second value of the thermistor current $I$ and so on. If we want the heater power as the parameter, we must also measure the heater voltage $U_h$ and adjust at each measurement the heater current $I_h$ so that $N_h = U_h I_h =$ const. From the measured values of $U$ and $I$ we can directly plot the $U$-$I$-characteristic and, by the aid of (1) and (2), also the $R$-$N$-characteristic. The case $N_h =$ const. also gives us the variation of $R_h = U_h/I_h$.

b. *Simplifying assumptions. Heater efficiency*

Unfortunately the task of determining a series of static characteristics like those in Figs. 4 and 5 is quite time-consuming. Good accuracy requires many measuring points and in going from measuring point to measuring point one has, on account of the thermal inertia of the thermistor, to wait a considerable time for the steady state. Every possibility to reduce the number of measuring points is therefore of value. In this respect we find great help in the following schematic picture of the thermistor, similar to the picture of the directly heated thermistor which has served as a basis for the earlier papers of this series:

*All parts of the thermistor body proper, i. e., the thermistor bead with its insulating cover and that part of the heater which is in immediate contact with the cover, have one and the same temperature $\Theta$. The remaining parts of the thermistor, i. e., connecting wires, glass bulb etc. have the temperature $\Theta_0$ of the surrounding medium.*

From these assumptions follows that, at a given temperature $\Theta_0$, the power $P$, dissipated as heat from the thermistor body to the surroundings, is uniquely determined by the temperature $\Theta$. The resistance $R$ of the thermistor bead being also uniquely determined by $\Theta$, we must have a one-to-one relation between $R$ and $P$. This relation is the resistance-power-characteristic of the thermistor. Hence, to each value of the resistance $R$ always corresponds one and

the same value of the dissipated heat power $P$. It follows that $P$ is constant on any straight line through the origin of the $U$-$I$-diagram (resistance line, see Fig. 4); it is also constant on a horizontal line in the $R$-$N$-diagram (Fig. 5). It should be observed that the $R$-$N$-characteristics are static while the $R$-$P$-characteristics are also valid under transient conditions.

In the stationary state the thermistor body is in thermal equilibrium. This implies that the dissipated heat power $P$ equals the *total* electric power supplied to the thermistor. This total power consists of two parts. From the thermistor circuit the electric power $N$ is brought to the thermistor body. But the thermistor body also becomes heated from the heater circuit. Of the entire heater power $N_h$, however, a certain amount is lost in those parts of the heater which have no direct contact with the thermistor body. Only a certain part $\eta N_h$ of $N_h$ is brought to the thermistor body. *The factor $\eta < 1$ is of paramount importance in our theory.* We will call $\eta$ the *heater efficiency.*

Thus, the *steady state* power balance equation of the indirectly heated thermistor can be written

$$P = N + \eta N_h \tag{5}$$

If the approximations, on which we have founded our discussion, are physically reasonable, we should expect a constant value of the heater efficiency. The practical importance of this conclusion calls for a thorough investigation of its validity. This investigation, reported in Chapter 5, shows that, for the thermistor types investigated, the heater efficiency is practically constant so that it does not depend on the operating conditions of the thermistor. We hence put

$$\eta = \text{constant} \tag{6}$$

We can compute $\eta$ from two pairs of power values, $(N_1, N_{h1})$ and $(N_2, N_{h2})$, corresponding to one and the same value of $P$ (and of $R$). Eq. (5) gives us

$$N_1 + \eta N_{h1} = N_2 + \eta N_{h2} \tag{7}$$

from which

$$\eta = -\frac{N_2 - N_1}{N_{h2} - N_{h1}} \tag{8}$$

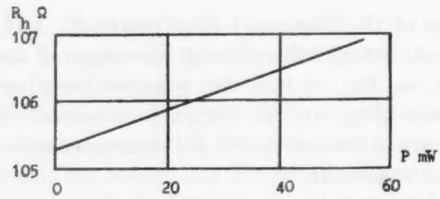



*Fig. 6.* Approximately linear relation between $R_h$ and $P$ for the thermistor of Fig. 4.

Going to the limit of small changes in $N$ and $N_h$, we find from eq. (8)

$$\eta = -\left(\frac{dN}{dN_h}\right)_{R=\text{constant}} \tag{9}$$

This expression can be used as a general definition of the heater efficiency, valid also if $\eta$ varies. In words:

*The heater efficiency is equal to the ratio between small changes in thermistor power and heater power chosen so that the thermistor resistance remains unaltered.*

On account of the fact that we have found $\eta$ = constant, we need not determine all the $U$-$I$- or $R$-$N$-characteristics through direct measurements. It is sufficient to determine $\eta$ and one single characteristic, the $R$-$P$-characteristic of the thermistor. Eq. (5) shows that $P = N$ for $N_h = 0$. Hence, the important $R$-$P$-characteristic is identical with that static $R$-$N$-characteristic which corresponds to $N_h = 0$.

We have mentioned earlier that if we want to make very accurate calculations, we must also know how the heater resistance $R_h$ varies with the operating conditions of the thermistor. Our schematic picture of the thermistor implies that that part of the heater which is in direct thermal contact with the thermistor bead has the temperature $\Theta$ of the bead, while the rest of the heater has the temperature of the surroundings. This means that we should have a one-to-one relation between $R_h$ and $\Theta$ and thus also between $R_h$ and $P$. The investigation in Chapter 5 shows that such a relation between $R_h$ and $P$ can be assumed even if we aim at high accuracy (see *Fig. 6*). The relation being linear, its construction only requires that we know two points on the $R_h$-$P$-characteristic.

c. *The resistance equation*

In Part 2 of this study it has been shown that the $R$-$P$-characteristic of a *directly* heated thermistor can be represented analytically with good accuracy by the *resistance equation*

$$R = R_\infty \, e^{\frac{B}{CP + \Theta_0}} \tag{10}$$

$R_\infty$ (unit: ohm) and $B$ (unit: °K) are characteristic constants of the thermistor in question, while the *power sensitivity* $C$ (unit: °K/mW) also depends on the properties of the surrounding medium. Now, we know that the $R$-$P$-characteristic of the indirectly heated thermistor is obtained as the relation between resistance and thermistor power with the heater power $= 0$. An indirectly heated thermistor without heater power being essentially equivalent to a directly heated thermistor, it can be surmised that eq. (10) should be valid also for the indirectly heated thermistor. For the thermistors investigated in Chapter 5 this proves to be the case.

By introducing (5) in (10) we obtain the expression of the static $R$-$N$-characteristics:

$$R = R_\infty \, e^{\frac{B}{CN + C\eta N_h + \Theta_0}} \tag{11}$$

Thus, the heater power $N_h$ has the same influence on the static characteristics as an increase of the ambient temperature by the amount $C\eta N_h$.

It has also been shown in Part 2 how the three constants $R_\infty$, $B$, and $C$ can be calculated. The simplest method is the following. Starting with three points $(R_1, P_1)$, $(R_2, P_2)$, and $(R_3, P_3)$ on the $R$-$P$-characteristic, we obtain from eq. (10) the two relations

$$\ln R_1 - \ln R_2 = \frac{BC\,(P_2 - P_1)}{(CP_1 + \Theta_0)\,(CP_2 + \Theta_0)} \tag{12 a}$$

$$\ln R_2 - \ln R_3 = \frac{BC\,(P_3 - P_2)}{(CP_2 + \Theta_0)\,(CP_3 + \Theta_0)} \tag{12 b}$$

which give us

$$C = -\,\Theta_0 \frac{(P_2 - P_3)\ln R_1 + (P_3 - P_1)\ln R_2 + (P_1 - P_2)\ln R_3}{P_1(P_2 - P_3)\ln R_1 + P_2(P_3 - P_1)\ln R_2 + P_3(P_1 - P_2)\ln R_3} \tag{13}$$

Introducing this value of $C$ into (12 a) or (12 b), we obtain $B$. $R_\infty$ then follows from (10) if we make use of one of the three known points, for instance $(R_1, P_1)$.

From what has been said, we can infer that the problem of determining the characteristics of the indirectly heated thermistor can be considerably simplified. In fact, the following measurements are sufficient:

1) The $R$-$P$-characteristic is determined as the relation between resistance and dissipated thermistor power with the heater power $= 0$. If we use the resistance equation (10), we only need three points on the $R$-$P$-characteristic in order to know it completely.

2) The heater efficiency is determined from two pairs of corresponding values of thermistor power and heater power, belonging to the same resistance value.

3) The linear relation between $R_h$ and $P$ is determined by measuring $R_h$ for two different values of $P$.

These measurements give us all the information about the indirectly heated thermistor which we need for the treatment of static problems. From the data they yield, we can compute in a simple manner the $R$-$N$-characteristics as well as the $U$-$I$-characteristics with $N_h$ as a parameter. $R_h$ being known, we can also obtain the characteristics with the parameter $I_h$.

d. *Dynamic resistance, dynamic factor*

The small signal equivalent circuit, deduced in Chapter 4, makes use of the *dynamic resistance* $r$ and the *dynamic factor* $F$ of the thermistor. Both quantities are a measure of the derivative of the static characteristic in a given operating point. The following expressions are generalizations of the corresponding definitions in Part 1:

$$r = \left(\frac{dU}{dI}\right)_{N_h = \text{constant}} \tag{14}$$

$$F = -\left(\frac{dR/R}{dN/N}\right)_{N_h = \text{constant}} = -\frac{N}{R}\left(\frac{dR}{dN}\right)_{N_h = \text{constant}} \tag{15}$$

Thus, the dynamic resistance is equal to the derivative of the static $U$-$I$-characteristic with $N_h$ as a parameter, while the dynamic factor is proportional to the derivative of the static $R$-$N$-characteristic with $N_h$ as a parameter.

Now, the two characteristics depend on each other. In fact, we have the following relation between $r$ and $F$, already derived in Part 1:

$$F = \frac{R - r}{R + r} \tag{16}$$

This can also be written

$$r = R \frac{1 - F}{1 + F} \tag{17}$$

In the maximum point of the static $U$-$I$-characteristic we have $r = 0$. It follows from (16) that in the maximum point $F = 1$; to the left of the maximum point we have $r > 0$, $F < 1$; to the right of it we have $r < 0$, $F > 1$. In the origin we have $F = 0$ and $r = R$.

In eq. (15) we have $N_h = \text{const.}$ Using (5) and (6) we can therefore write

$$F = -\frac{N}{R}\frac{dR}{dP} \tag{18}$$

or

$$F = -Nm \tag{19}$$

where we have introduced the *power coefficient* $m$ of the resistance:

$$m = \frac{1}{R}\frac{dR}{dP} \tag{20}$$

The power coefficient is equal to the ratio between a certain relative change of the resistance and the corresponding absolute change of the dissipated heat power.

By the aid of eq. (10) we can obtain an analytical expression of the dynamic factor $F$, which obviates the need for graphic determinations of derivatives. Taking the logarithm of (10), and using (18), we immediately obtain

$$F = \frac{BCN}{(CP + \Theta_0)^2} \tag{21}$$

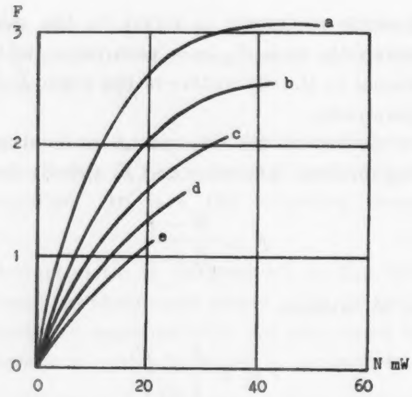



*Fig. 7.* Dynamic factor $F$ as a function of $N$. Same thermistor as in Fig. 4.

a) $N_h = 0$ d) $N_h = 30$ mW
b) $N_h = 10$ mW e) $N_h = 40$ mW
c) $N_h = 20$ mW

or, after introducing (5):

$$F = \frac{BCN}{(CN + C\eta N_h + \Theta_0)^2} \tag{22}$$

It follows that the dynamic factor at constant heater power has a maximum value

$$F_{\max} = \frac{B}{4\,(C\eta N_h + \Theta_0)} \tag{23}$$

occurring for

$$N = \eta N_h + \Theta_0/C \tag{24}$$

We see from (23) that the largest possible value of $F$ is $B/4\Theta_0$, occurring when $N_h = 0$ and $N = \Theta_0/C$. For the thermistors investigated numerically in this paper, the constant $B$ is of the order of magnitude 3 600° K (high resistance thermistors) or 2 400° K (low resistance thermistors). Hence, at room temperature (300° K) the maximum values of $F$ are of the order of magnitude 3 and 2. For the thermistor of Fig. 4, *Fig. 7* gives the relation between $F$ and $N$ with $N_h$ as a parameter.

When $F$ has been determined from (22), $r$ follows at once from (17). *Fig. 8* gives the relation $r$ vs. $I$ with $N_h$ as a parameter.

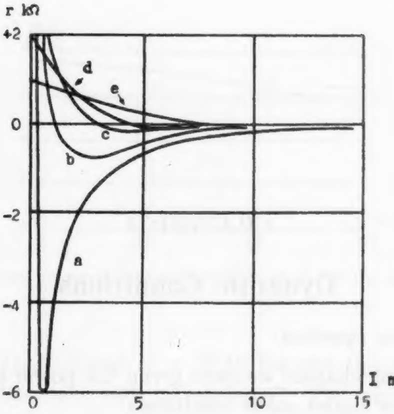



*Fig. 8.* Dynamic resistance $r$ as a function of $I$. Same thermistor as in Fig. 4.

a) $N_h = 0$ d) $N_h = 30$ mW
b) $N_h = 10$ mW e) $N_h = 40$ mW
c) $N_h = 20$ mW

A comparison with the earlier papers of this series shows that our present formulae are immediately applicable to a directly heated thermistor, if we put $N_h = 0$.

Inversely, on account of the equivalence between an increase of the heater power and an increase of the ambient temperature, the earlier results are directly applicable to the indirectly heated thermistor. We only have to replace everywhere the ambient temperature $\Theta_0$ with the "equivalent ambient temperature" $\Theta_0 + C\eta N_h$.

N. B. For a directly heated thermistor in the static case it is not necessary to distinguish between $P$ and $N$. Our earlier formulae therefore sometimes have $P$ where the present paper has $N$.

## CHAPTER 3

# Dynamic Conditions

a. *The thermistor equation*

In the preceding chapter we have given the power balance equation of the thermistor under *static conditions:*

$$N + \eta N_h = P \tag{5}$$

Under dynamic conditions part of the supplied electric power serves to increase the *thermal energy* $W$ of the thermistor body. Therefore, in this case the power balance equation reads

$$N + \eta N_h = P + \frac{dW}{dt} \tag{25}$$

In words: supplied electric power = dissipated heat power + increase per unit time of the thermal energy.

As in the theory of the directly heated thermistor we introduce the *thermal time constant* $T$ of the thermistor,

$$T = \frac{dW}{dP} \tag{26}$$

Then (25) can be written

$$N + \eta N_h = P + T\frac{dP}{dt} \tag{27}$$

This is the *thermistor equation*, generalized to the case of the indirectly heated thermistor. As shown in Parts 1 and 2, $T$ varies slightly with the temperature $\Theta$ and thus also with $P$. Hence, in principle, $T$ in eq. (27) is not a constant but a function of $P$, $T = T(P)$.

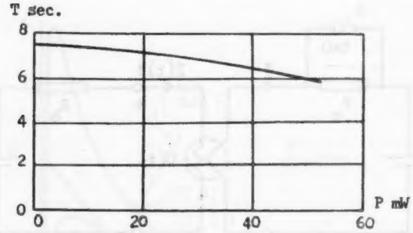



*Fig. 9.* Thermal time constant $T$ as a function of $P$ for the thermistor of Fig. 4.

*Fig. 9* gives the relation $T = T(P)$ for the thermistor of Fig. 4. We see that $T$ decreases with increasing $P$. The variations are small, however, so that it is often possible to put $T =$ const.

Furthermore, the approximation involved in our assumption of a homogeneous temperature of the thermistor body makes it reasonable to neglect variations in $T$. In fact, especially under transient conditions only moderate accuracy can be expected from the thermistor equation. The heat transport within the thermistor body is not instantaneous, and hence a sudden increase of the heater power cannot immediately influence the temperature of the thermistor bead. For the same reason, a decrease of the thermistor power gives at the beginning of the transient period a higher temperature to the thermistor bead than the one deduced from our simplified theory.

For the directly heated thermistor it has been shown in Parts 2 and 3 that the uneven temperature distribution occasionally gives large deviations between theory and actual phenomena. If the power changes are not too large, however, the accuracy of the thermistor equation is sufficient. Hence, it does not seem necessary to develop a theory, which takes into account the uneven temperature distribution.

**b.** ***Determination of the thermal time constant***

Dynamic problems can be solved by a direct use of the thermistor equation (27). If the thermistor current is so small that we can put $N \approx 0$, the calculations become especially simple. In such cases the thermistor equation can be written

$$\eta N_h = P + T \frac{dP}{dt} \tag{28}$$

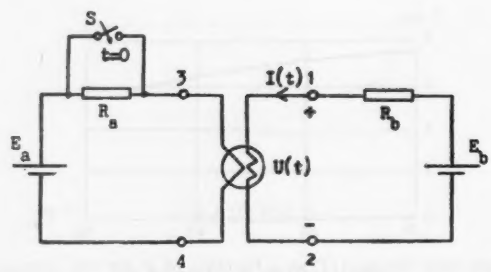



*Fig. 10.* Circuit for measuring the thermal time constant of an indirectly heated thermistor.

With $N_h(t)$ known, we compute $P(t)$ by direct integration and then obtain $R(t)$ from the $R$-$P$-characteristic. Let us suppose that $N_h$ makes a sudden step change from the value $N_{h1}$ to the value $N_{h2}$. Then eq. (28) gives us

$$P = \eta N_{h1} + \eta\,(N_{h2} - N_{h1})\,(1 - e^{-t/T}) \tag{29}$$

Thus, the dissipated heat power varies exponentially with the time constant $T$ from the value $P_1 = \eta N_{h1}$ to the value $P_2 = \eta N_{h2}$. This result gives us a simple experimental method for determining the thermal time constant, based on the circuit of *Fig. 10.*

In this circuit the heater is connected in series with a resistance $R_a$, which can be short-circuited by a switch $S$. The resistance $R_a$ should be much smaller than $R_h$ so that the closing of $S$ gives a very small change of $N_h$. Then the thermistor resistance $R$ also changes very little and we can assume a linear relation between $R$ and $P$. It then follows from (29) that $R$ decreases exponentially with the time constant $T$. The thermistor voltage $U$ also decreases and for small changes this decrease will be proportional to the decrease of the resistance. Hence, $U$ also changes exponentially with the time constant $T$ so that we can obtain $T$ from a record of $U$. (Of course, $I$ and $U$ must be so small that the condition $N \approx 0$ is fulfilled.) By determining $T$ for a number of different operating points we find how $T$ depends on $P$.

c. *Application of the thermistor equation*

An example will be given which shows how the thermistor equation (27) can be used for solving a dynamic problem. Let us try to cal-

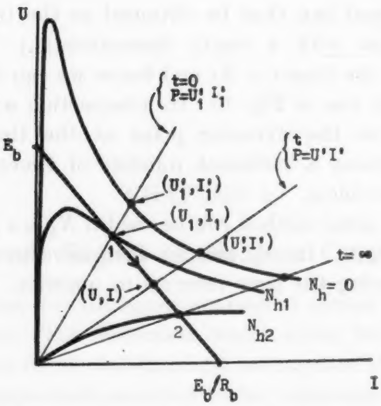



***Fig. 11.*** **To the calculation of the transients in the circuit, Fig. 10.**
$N_{h1} = E_a^2 R_h/(R_h + R_a)^2$; $N_{h2} = E_a^2/R_h$

**culate $U(t)$ and $I(t)$ in the circuit of Fig. 10, assuming the general case where the power $N$ cannot be neglected.**

**The heater power is:**

$$\text{for } t < 0 : N_{h1} = E_a^2 R_h/(R_h + R_a)^2$$

$$\text{for } t > 0 : N_{h2} = E_a^2/R_h$$

**We assume $R_h =$ constant. *Fig. 11* gives the three static $U$-$I$-characteristics, corresponding to the heater powers $N_{h1}$, $N_{h2}$, and $N_h = 0$. When the heater power jumps from $N_{h1}$ to $N_{h2}$, the operating point $(U, I)$ moves along the dynamic characteristic from point *1* to point *2*. Our problem is solved if in every moment we know the position on the dynamic characteristic of the dynamic point. We start at $t = 0$ in the operating point $(U_1, I_1)$ and compute the operating point at $t = \Delta t \ll T$. From (27) follows**

$$\Delta P = \frac{\Delta t}{T} (N + \eta N_{h2} - P) \tag{30}$$

**Here we put $N = N_1 = U_1 I_1$ and $P = P_1 = U_1' I_1'$, where $(U_1', I_1')$ is the intersection between the resistance line through point *1* and the static $U$-$I$-characteristic for $N_h = 0$. ($P$ has a constant value on**

a resistance line and can thus be obtained as the thermistor power at its intersection with a static characteristic.) We now know $P = P_1 + \Delta P$ at the time $t = \Delta t$ and hence we can draw the corresponding resistance line in Fig. 10. Its intersection with the dynamic characteristic gives the dynamic point at the time $t = \Delta t$. By repeating the process a sufficient number of times we obtain the solution of our problem.

Essentially the same method can be used if $N_h$ is a general function of time, $N_h = N_h(t)$. In eq. (30) we then have to use the average value of $N_h(t)$ during the time interval in question.

## CHAPTER 4

# Small Signal Equivalent Circuit

We shall deduce a small signal equivalent circuit for the indirectly heated thermistor. The deduction, which starts from the thermistor equation, is based on the fundamental assumption that the variations of thermistor temperature and thermistor resistance are so small that in performing series developments around an operating point we need only consider the linear terms. We further neglect variations in $T$ and $R_h$, assuming these quantities to have constant values, determined by the position of the operating point.

We introduce the symbols, given below in (31) and (32). In these relations $u, i, \Delta R$ etc. are deviations from the values $U_0, I_0, R_0$ etc. in the operating point, where the thermistor is in thermal balance.

$$\left.\begin{aligned} U &= U_0 + u \\ I &= I_0 + i \\ R &= R_0 + \Delta R \end{aligned}\right\} \quad (31) \qquad \left.\begin{aligned} N &= N_0 + \Delta N \\ N_h &= N_{h0} + \Delta N_h \\ P &= P_0 + \Delta P \end{aligned}\right\} \quad (32)$$

The values in the operating point satisfy the relations

$$R_0 = U_0/I_0 \tag{33}$$

$$N_0 = U_0 I_0 \tag{34}$$

Introducing (32) and (34) in the thermistor equation (27), we obtain

$$\Delta N + \eta \Delta N_h = \Delta P + T \frac{d}{dt} (\Delta P) \tag{35}$$

Making use of the power coefficient $m$, defined in (20), we can put

$$\Delta P = \frac{1}{m} \frac{\Delta R}{R_0} \tag{36}$$

Here we assume that

$$\Delta R \ll R_0 \tag{37}$$

Introducing (36) in (35) we obtain

$$\Delta N + \eta \Delta N_h = \frac{1}{mR_0}\left[\Delta R + T\,\frac{d}{dt}\,(\Delta R)\right] \tag{38}$$

Evidently, the condition (37) also implies that the change in total supplied power, $\Delta N + \eta \Delta N_h$, should be small compared to $P_0$. The relative changes $\Delta N/N_0$ and $\Delta N_h/N_{h0}$, however, are not necessarily small. If, for instance, the heater power increases from $N_{h0} = 0$ to a value $\Delta N_h \ll P_0$, we obtain an infinite relative change in heater power, but eq. (38) is still valid.

In order to obtain a practically manageable theory, we now assume that the variations in current and voltage on the thermistor side are small. It follows that we can put

$$\left.\begin{aligned} \Delta N &= N_0\left(\frac{u}{U_0} + \frac{i}{I_0}\right) \\ \Delta R &= R_0\left(\frac{u}{U_0} - \frac{i}{I_0}\right) \end{aligned}\right\} \tag{39}$$

We now introduce (39) into (38) and go over to operational expressions by putting $p = \dfrac{d}{dt}$. If we retain the time function symbols $u$, $i$, $\Delta N_h$ as symbols for the corresponding operators, we obtain

$$N_0\left(\frac{u}{U_0} + \frac{i}{I_0}\right) + \eta \Delta N_h = \frac{1}{m}\left(\frac{u}{U_0} - \frac{i}{I_0}\right)(1 + pT) \tag{40}$$

Introducing in (40) $F = -mN_0$ according to (19) and simplifying, we find

$$u - iR_0\,\frac{1 - F + pT}{1 + F + pT} + \frac{F}{I_0}\,\frac{\eta \Delta N_h}{1 + F + pT} = 0 \tag{41}$$

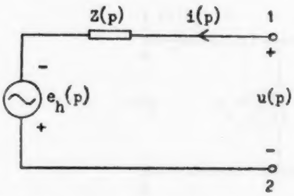



*Fig. 12.* Circuit for which equation (41) is valid.

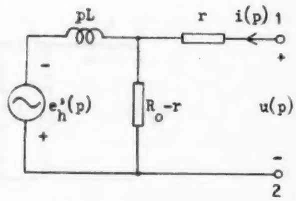



*Fig. 13.* Circuit, having the circuit of Fig. 12 as its Thévenin equivalent.

Evidently, this relation is valid for the circuit of *Fig. 12* if in this circuit we have

$$Z\,(p) = R_0\,\frac{1 - F + pT}{1 + F + pT} \tag{42}$$

$$e_h\,(p) = \frac{F}{I_0}\,\frac{\eta \Delta N_h(p)}{1 + F + pT} \tag{43}$$

It is easy to see that the circuit of Fig. 12 can be obtained through an application of Thévenin's theorem to the circuit of *Fig. 13,* if in this latter circuit we have

$$r\; = R_0\,\frac{1 - F}{1 + F} \tag{17}$$

$$L = R_0 T\,\frac{2\,F}{(1 + F)^2} \tag{44}$$

$$e'_h\,(p) = e_h\,(p)\,(1 + F + pT)/(1 + F) \tag{45}$$

Introducing eq. (43) into (45), we obtain

$$e'_h\,(p) = \frac{F}{1 + F}\,\frac{\eta \Delta N_h(p)}{I_0} \tag{46}$$

The power $\Delta N_h$ is a quadratic function of current and voltage and therefore somewhat inconvenient for our calculations. Provided that $\Delta N_h \ll N_{h0}$ we can, however, write $\Delta N_h$ as a linear expression. We have $N_h = I_h^2 R_h$ and hence, for small relative changes in $N_h$,

$$\Delta N_h = 2\,I_{h0} R_h i_h = 2\,I_{h0} u_h \tag{47}$$

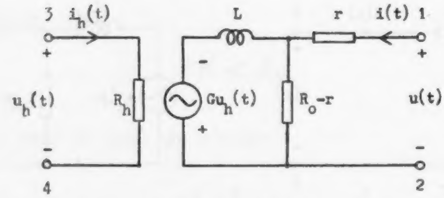



*Fig. 14.* Equivalent circuit of an indirectly heated thermistor for small superimposed currents and voltages.

Here $i_h$ and $u_h$ are small deviations in heater current and heater voltage from the values $I_{h0}$ and $U_{h0}$ in the operating point. Introduction of (47) in (46) gives us the final expression

$$e_h'(p) = \frac{2F}{1+F}\,\frac{\eta\, I_{h0}}{I_0}\,u_h(p) \tag{48}$$

We see that besides the operators $e_h'(p)$ and $u_h(p)$, eq. (48) only contains constant quantities, determined by the position of the operating point. We can therefore make a direct translation from the operators to the corresponding time functions.

Hence, *Fig. 14* gives an equivalent circuit of the indirectly heated thermistor for small superimposed voltages and currents in heater and thermistor circuit. The quantities $r$, $L$, and $G$ are given by the following expressions:

$$\boxed{\begin{aligned} r &= R_0\,\frac{1-F}{1+F} \\ L &= R_0 T\,\frac{2F}{(1+F)^2} \\ G &= \frac{2F}{1+F}\,\frac{\eta\, I_{h0}}{I_0} \end{aligned}} \tag{49}$$

We see that in the equivalent circuit the small heater voltage change appears as a small emf in series with the inductance, equal to the heater voltage change times a constant $G$. The value of

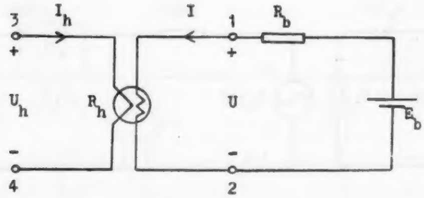



*Fig. 15.* Simple circuit studied by the aid of the small signal equivalent circuit of Fig. 14.

this constant is determined by the position of the operating point only. For constant heater conditions we have $u_h = 0$ and come back to the equivalent circuit of the directly heated thermistor as deduced in Part 1.

If the heater power in the operating point is so small that the condition $\Delta N_h \ll N_{h0}$ is *not* fulfilled, we cannot apply eq. (47) or use the circuit of Fig. 14. It is easy to see, however, that in this case we only have to replace the emf $Gu_h(t)$ with an emf $G'\eta\Delta N_h(t)$, where $G' = \dfrac{F}{1+F}\,\dfrac{1}{I_0}$.

*Application*

Let us assume that we have connected an indirectly heated thermistor to a resistive electric network. By the aid of Thévenin's theorem we can always reduce the network to a series combination of an emf $E_b$ and a resistance $R_b$, *Fig. 15*. Using the equivalent circuit of *Fig. 16*, we can calculate how the thermistor voltage $U$ changes at small variations $u_h(t)$ of the heater voltage $U_h$. We assume that the emf $E_b$ is constant. Applying Thévenin's theorem to the equi-

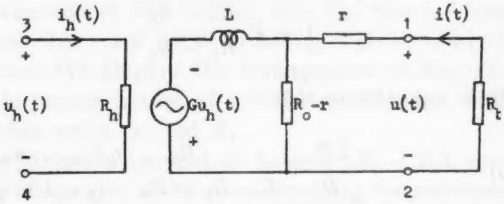



*Fig. 16.* Equivalent circuit of the arrangement, Fig. 15.

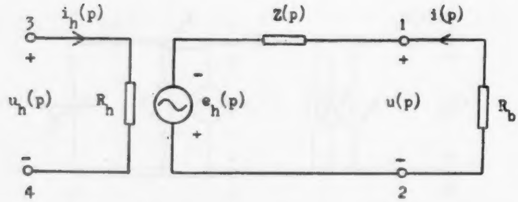



*Fig. 17.* The Thévenin equivalent of the circuit, Fig. 16.

valent circuit, we obtain the circuit of *Fig. 17*, where the time functions have been replaced by operators. The impedance $Z(p)$ is given by eq. (42). The emf $e_h(p)$ follows directly from (43) and (47):

$$e_h(p) = 2\,F\,\frac{\eta\, I_{h\,0}}{I_0}\,\frac{u_h\,(p)}{1 + F + pT} \tag{50}$$

Further

$$i(p) = \frac{e_h(p)}{Z(p) + R_b} \tag{51}$$

and

$$u(p) = -\,i(p)\,R_b \tag{52}$$

Making use of (42) we find after some calculations

$$Z(p) + R_b = \frac{1 + F\,\dfrac{R_b - R_0}{R_b + R_0} + pT}{1 + F + pT}\,(R_b + R_0) \tag{53}$$

Introducing the *effective time constant* of the thermistor

$$\tau = \frac{T}{1 + F\,\dfrac{R_b - R_0}{R_b + R_0}} \tag{54}$$

we obtain from eqs. (50) to (53)

$$u(p) = -\,\frac{2\,F}{1 + F\,\dfrac{R_b - R_0}{R_b + R_0}}\,\frac{R_b}{R_b + R_0}\,\frac{\eta\, I_{h\,0}}{I_0}\,\frac{u_h}{1 + p\,\tau} \tag{55}$$

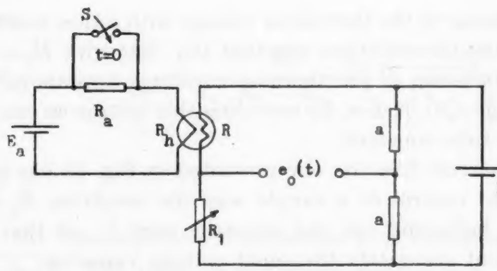



*Fig. 18.* **Another circuit (cf. Fig. 10) for measuring the thermal time constant of an indirectly heated thermistor.**

**Two cases are of special interest, viz., a) $u_h(t)$ changes stepwise, b) $u_h(t)$ changes sinusoidally.**

**a) $u_h(t) =$ *a sudden step with the height* $u_h$**

**From (55) we obtain directly**

$$u(t) = -\frac{2F}{1 + F\dfrac{R_b - R_0}{R_b + R_0}} \frac{R_b}{R_b + R_0} \frac{\eta I_{h0}}{I_0} u_h (1 - e^{-t/\tau}) \quad (56)$$

**The voltage follows an exponential law with the time constant $\tau$.**

**Eq. (54) gives us**

$$\left.\begin{array}{l} R_b > R_0 : \tau < T \\ R_b = R_0 : \tau = T \\ R_b < R_0 : \tau > T \end{array}\right\} \quad (57)$$

**Thus, when the external resistance is = the thermistor resistance, the time constant of the voltage is = the thermal time constant.**

**We can use this result for determining experimentally the thermal time constant. We employ the arrangement of *Fig. 18.* Here the thermistor forms one branch of a Wheatstone bridge, while the three other branches are $a$, $a$, and $R_1$.**

**First the bridge is brought to balance ($R_1 = R_0$), so that it has the output voltage $e_0 = 0$. By short-circuiting the resistance $R_a \ll R_h$, we then bring about a small change of the heater voltage. This gives**

us a small change of the thermistor voltage with a time constant which is equal to the thermal time constant (eq. (54) with $R_b = R_1 = R_0$). The entire variation of the thermistor voltage appears as the bridge output voltage $e_0(t)$. Hence, by recording this voltage we can determine the thermal time constant.

A bridge circuit like the one presented in Fig. 18 has two advantages: we can control in a simple way the condition $R_b = R_0$, and we can, by balancing out the constant part $U_0$ of the thermistor voltage, record accurately the small voltage variation.

By the aid of the equivalent circuit it is easy to show that we have the same time constant for the transient response to a small change of the emf on the thermistor side ($E_b$ in Fig. 15) as for the transient response to a small change of the heater voltage. We can therefore, by short-circuiting a small resistance in series with the emf which feeds the bridge, also determine $T$ from a small change on the thermistor side. We then come back on a method used in Part 2 for determining the thermal time constant of the directly heated thermistor. In Part 2 the errors of this method are discussed.

In Chapter 3 we gave another method for the determination of the thermal time constant and verified it by a qualitative reasoning, based on the thermistor equation. We can now base the verification directly on eq. (56). We have assumed that the thermistor power can be neglected as compared to the heater power. Hence, $F \ll 1$, so that (56) can be written

$$u\,(t) = -\,2\,F\,\frac{R_b}{R_b + R_0}\,\frac{\eta\, I_{h0}}{I_0}\,u_h\,(1 - e^{-t/\tau}) \tag{58}$$

According to (54) we have

$$\tau = T \tag{59}$$

and thus the thermal time constant is obtained from a record of $u(t)$.

b) $u_h(t) =$ *a sinusoidal voltage*

Putting $u_h(t) = \hat{u}_h \sin \omega t$, we obtain the thermistor voltage variation $u(t) = \hat{u} \sin (\omega t + \varphi)$ by replacing in (55) the operator $p$ by $j\omega$:

$$\hat{u}\, e^{j\varphi} = -\,\frac{2\,F}{1 + F\,\dfrac{R_b - R_0}{R_b + R_0}}\,\frac{R_b}{R_b + R_0}\,\frac{\eta\, I_{h0}}{I_0}\,\frac{\hat{u}_h}{1 + j\omega\tau} \tag{60}$$

It follows

$$\hat{u} = \frac{2F}{1 + F\,\dfrac{R_b - R_0}{R_b + R_0}} \, \frac{R_b}{R_b + R_0} \, \frac{\eta I_{h0}}{I_0} \, \frac{\hat{u}_h}{\sqrt{1 + (\omega\tau)^2}} \tag{61 a}$$

$$\varphi = \pi - \arctan \omega\tau \tag{61 b}$$

We see that the amplitude $\hat{u}$ decreases with increasing frequency as $1/\sqrt{1 + (\omega\tau)^2}$. The value of the phase angle is $\varphi \approx 180°$ at very low frequencies and decreases with increasing frequency. At very high frequencies it approaches $90°$. For $\omega\tau = 1$ the amplitude is $1/\sqrt{2}$ times its value at very low frequencies while the phase angle is $135°$.

CHAPTER 5

# Experiments

The measurements described below demanded highly stable thermistor surroundings. In particular, the surrounding medium had to be kept at a very constant temperature. The thermistor was therefore enclosed in a metal cylinder, filled with paraffin oil and immersed in a temperature controlled bath.

For some measurements this bath was an oil bath, maintained at 25° C with a constancy of $\pm$ 0.01° C by a thermostat with a contact thermometer. Particularly for dynamic measurements even such small fluctuations turned out to cause unpermissible movements of the operating point of the thermistor. Hence, for some measurements the thermostat consisted of an ice bath. The metal cylinder with the thermistor was immersed in a one litre Dewar flask, filled with melting ice made from distilled water. The Dewar flask was placed in a cardboard box and thermally insulated with paper.

The dynamic phenomena were so slow that they could be recorded on an $xy$-recorder (Moseley Autograf, Model 1). The time axis was obtained by feeding the $x$-amplifier with the output voltage from a linear potentiometer, driven by a constant speed motor.

For some of the measurements use was made of an ultralow-frequency generator, constructed according to *Fig. 19.* Its principles are as follows. The rotor of a resolver is driven with constant angular speed $\omega$ by a motor working over a gear box. Electrically, the rotor is fed from a 1 000-cycle A. C. voltage. Through electromagnetic induction this voltage gives rise in the stator to a sinusoidally modulated 1 000-cycle A. C. voltage with a modulation frequency = the rotor speed (turns per second). In the resistance $R_4$ a small part of the stator voltage is added to a small part of the given 1 000-cycle voltage. Hence, the input voltage to the amplifier can be written as $e' = A\,(1 + k \sin \omega t) \sin 2\pi \cdot 1000\,t$. The amplitude $A$ is regulated with

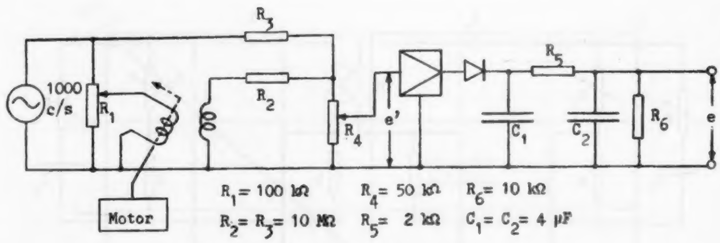



*Fig. 19.* Ultralow-frequency generator.

$R_4$, the modulation index $k$ with $R_1$. After amplification the signal is demodulated to an output voltage $e = E\,(1 + k \sin \omega t)$. A satisfactory demodulation requires $k < 0.7$ to $0.8$ and $\omega/2\pi \ll 1\,000$ c/s.

A sweep, synchronous with the signal, was obtained by driving the time axis potentiometer from the resolver axis with a gear ratio 1 : 2. Hence, one period of the signal always occupies one and the same length on the record, independently of the $\omega$-value.

A. *Static conditions*

The static characteristics of a number of indirectly heated thermistors have been accurately determined. These measurements have served a twofold purpose: to study the variation of the heater efficiency and to investigate the accuracy which can be expected from the resistance equation.

The heater efficiency is of particular interest. Therefore, a measuring method especially suitable for its determination was chosen. In Chapter 2 a general definition of the heater efficiency has been given:

$$\eta = -\left(\frac{dN}{dN_h}\right)_{R=\text{constant}} \tag{9}$$

Thus, $\eta$ can be determined from the curves giving $N$ as a function of $N_h$ with $R$ as a parameter.

The circuit of *Fig. 20* was employed for obtaining these curves. The heater with the resistance $R_h$ forms one branch of a Wheatstone bridge; the three resistors $a$, $a$, and $R_1$ form the other branches. From a battery $E_1$ the bridge is fed with a current $I_1$ which can be regulated with the resistor $R_2$. With the switch $S$ in position *1* the

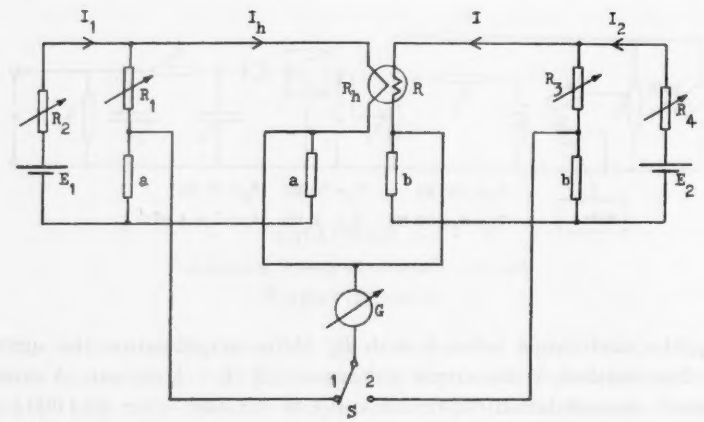


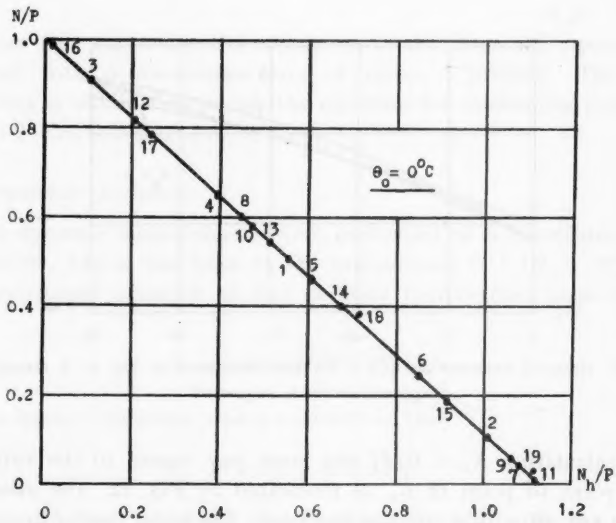

*Fig. 20.* **Arrangement for accurate determination of the static characteristics.**

bridge balance is indicated by the galvanometer $G$. At balance we have $R_h = R_1$, giving the heater current $I_h = \frac{1}{2} I_1$. The current $I_1$ can be accurately determined by the aid of a potentiometer (which is not shown in the diagram). The thermistor and the three resistors $b$, $b$, and $R_3$ form a second Wheatstone bridge, similar to the first. The second bridge is in balance for $R = R_3$ and $I = \frac{1}{2} I_2$. The current $I_2$ is also determined in the potentiometer.

In carrying through a measurement a certain constant value is first given to $R_3$. Then, with $I_1 = I_h = 0$, the value of the current $I$ which gives $R = R_3$ is determined (bridge 2 in balance). $I$ is then decreased and both bridges brought to balance simultaneously, bridge *1* by regulating $R_1$, and bridge *2* by varying the heater current. At the same time the two currents $I$ and $I_h$ are also determined.

A pair of corresponding values of the thermistor power $N = RI^2$ and the heater power $N_h = R_h I_h^2$ has now been obtained. The current $I$ is further diminished and a new pair of values $(N, N_h)$ is determined and so on. The measurements also display the variation of $R_h$.

An investigation of several different thermistor types gave entirely consistent values. *Fig. 21* shows the results for the thermistor of Figs. 4 to 9 (Stantel, type B 2552/60) at $\Theta_0 = 0°$ C. The diagram gives $N/P$ vs. $N_h/P$, where the dissipated heat power $P$ has been determined as the thermistor power corresponding to $N_h = 0$. Evi-




| Point | 1 - 2 | 3 - 7 | 8 - 9 | 10 - 11 | 12 - 15 | 16 - 19 |
|---|---|---|---|---|---|---|
| R kΩ | 100 | 40 | 10 | 4 | 1 | 0.4 |
| P mW | 5.97 | 9.96 | 17.39 | 23.57 | 35.96 | 47.23 |

*Fig. 21.* Relation between $N$, $N_h$, and $P$ for the thermistor of Fig. 4.

**dently, all measured points are very close to a straight line. Hence, we can write with good accuracy the relation between $N, N_h$, and $R$ as**

$$N + \eta\, N_h = P(R)$$

**with $\eta$ = constant and $P(R)$ = the thermistor power which for $N_h = 0$ gives the thermistor resistance $R$. From Fig. 21 we find $\eta = 1/1.135 = 0.881$.**

**Only two of the measured points, 18 and 19, deviate noticeably from the straight line. These points correspond to relatively high thermistor temperatures. All the other measured points show a deviation from the straight line which corresponds to a variation in $\eta$ of only a few tenths of a percent. This small variation is entirely covered by the measuring errors.**

**Measurements at $\Theta_0 = 25°$ C gave the same value of $\eta$ as those at $\Theta_0 = 0°$ C. This indicates that $\eta$ is also independent of the ambient temperature.**

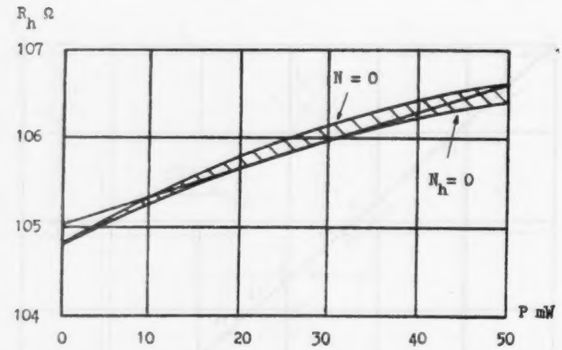



*Fig. 22.* Relation between $R_h$ and $P$ for the thermistor of Fig. 4. A straight line approximation is suggested.

In calculating $N_h = R_h I_h^2$ one must pay regard to the variation from point to point of $R_h$, as illustrated by *Fig. 22*. The measured points are all within the hatched part. The lower border line corresponds to $N_h = 0$, the upper border line to $N = 0$. Hence, strictly speaking, two different pairs of values $(N, N_h)$, which correspond to one and the same value of $P$, give somewhat different values of $R_h$. In other words, the relation between $R_h$ and $P = N + \eta N_h$ is not an exact one-to-one relation.

In fact, if we increase the heater power and, at the same time, decrease the thermistor power so as to keep $P$ constant, we obtain a small increase of $R_h$. The reason is that at large heater power and small thermistor power the *entire* heater wire becomes heated, while at small heater power and large thermistor power only that part of the heater wire becomes heated which is in direct contact with the thermistor body. In most cases this variation can be neglected so that the relation between $R_h$ and $P$ can be approximated by a straight line as indicated in Fig. 22.

The resistance equation (10) applies with excellent accuracy to the thermistors investigated. Consider, for instance, the thermistor B 2552/60, mentioned above. The table in Fig. 21 gives six points $(R, P)$ on the $R$-$P$-characteristic. From three of them (100 kohms; 5.973 mW), (10 kohms; 17.39 mW), and (1 kohm; 35.96 mW) we compute the three constants $R_\infty$, $B$, and $C$. This gives us the resistance equation

$$R = 0.630\, e^{\frac{3740}{6.53 P + \theta_0}} \text{ ohms}$$

From this expression the resistances in the three other points are derived with a maximum error of about 1 percent. The same accuracy is obtained if we use the equation for calculating points on the $R$-$P$-characteristic for $\Theta_0 = 25°$ C.

B. *Dynamic conditions*

The dynamic measurements were performed on a thermistor, type B 2321/60, which was kept at the temperature 0° C ($\Theta_0 = 273°$ K). The resistance equation at this ambient temperature came out as

$$R = 0.0931\, e^{\frac{2960}{5.58P + 273}} \text{ ohms}$$

The heater efficiency was $\eta = 0.862$ so that

$$P = N + 0.862\, N_h$$

*Figs. 23* and *24* show the $R$-$P$-characteristic and the static $U$-$I$-characteristics.

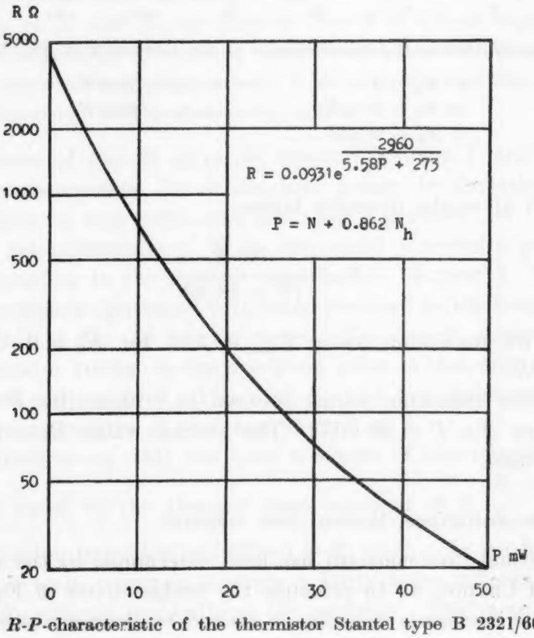



*Fig. 23.* $R$-$P$-characteristic of the thermistor Stantel type B 2321/60 at 0° C.

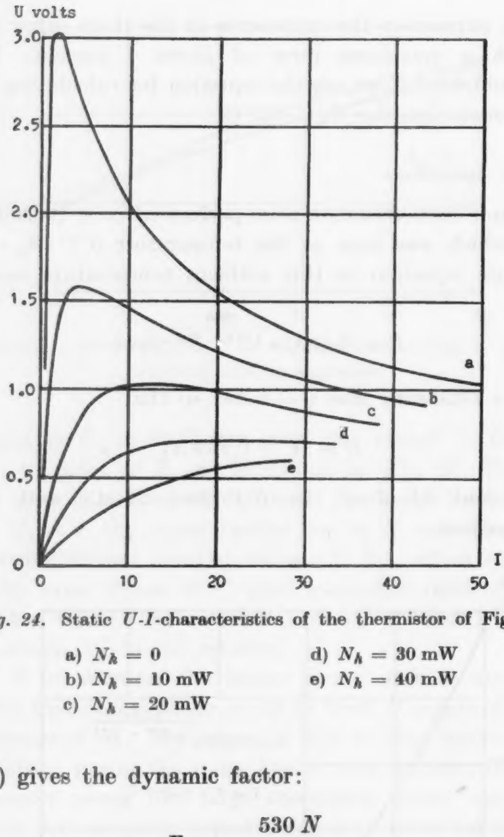



*Fig. 24.* **Static *U-I*-characteristics of the thermistor of Fig. 23.**

**a) $N_h = 0$**
**b) $N_h = 10$ mW**
**c) $N_h = 20$ mW**
**d) $N_h = 30$ mW**
**e) $N_h = 40$ mW**

**Eq. (21) gives the dynamic factor:**

$$F = \frac{530\, N}{(P + 48.9)^2}$$

**$F$ has its maximum value $F_{\max} = 2.71$ for $N_h = 0$, $N = P = 48.9$ mW.**

**The heater resistance varies between 98.8 ohms (for $P = 0$) and 101.2 ohms (for $P = 50$ mW). The average value $R_h = 100$ ohms has been used.**

**a) *Stepwise variations, thermal time constant***

**The thermal time constant has been determined by the aid of the method of Chapter 4. In principle the bridge circuit of Fig. 18 was used, where transients can also be excited by short-circuiting a small**

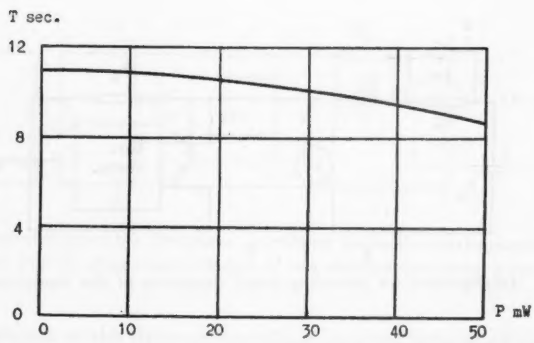



*Fig. 25.* Thermal time constant $T$ as a function of $P$ for the thermistor of Fig. 23.

resistance inserted in series with $E_b$. From about 30 measurements at different operating points the following conclusions have been drawn:

1) Changes on the thermistor side and on the heater side give the same time constant.
2) The thermal time constant is practically the same for different values of $N$ and $N_h$ as long as $P = N + \eta N_h$ is kept constant. There is possibly a tendency that $T$ increases with increasing $N_h$. The observed variations, about 5 percent, were of the same order of magnitude as the measuring errors.

The curve of *Fig. 25* gives the relation between $T$ and $P$ yielded by the measurements. In all measured points the deviation from the value given by this curve was less than 5 percent.

Some determinations of $T$ at very small thermistor powers were made according to the method described in Chapter 3. The results were in complete agreement with those obtained by the bridge method. The circuit of *Fig. 26* was employed. The emf $E_c$ compensates for the thermistor voltage in the operating point so that only the voltage variations are recorded. The emf $E_b$ and the resistance $R_b$ were chosen so as to give $N \approx 0$ and thus $F \ll 1$.

According to eq. (54) the time constant of the recorded voltage becomes equal to the thermal time constant if $F \dfrac{R_b - R_0}{R_b + R_0} \ll 1$. This condition is evidently fulfilled if $R_b \approx R_0$. As we have chosen $F \ll 1$, we can, however, tolerate a fairly large difference between $R_b$ and $R_0$ without jeopardizing the accuracy. This is an advantage

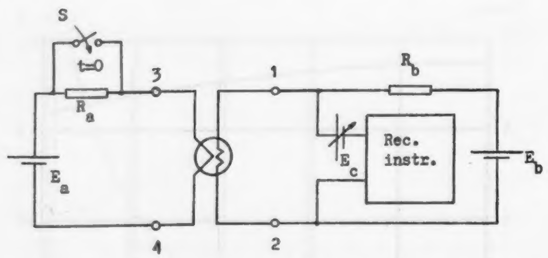



*Fig. 26.* Arrangement for recording small variations of the thermistor voltage.

against the bridge method where already a small difference between $R_b$ and $R_0$ gives a large error, so that it becomes necessary to correct for the influence of either the input impedance of the recording instrument or the resistance of the battery branch (cf. Part 2, p. 23). In most cases, therefore, the method of Fig. 26 is to prefer.

*Table 1* gives the result of measurements for such values of the series resistance $R_b$ (see Fig. 15) that the effective time constant $\tau$ of the thermistor differs considerably from the thermal time constant. The circuit of Fig. 26 was employed. Considering the many possible sources of error, the agreement with theory must be considered as satisfactory.

TABLE 1. Effective time constant $\tau$. Comparison between theoretical values from eq. (54) and experimental values, obtained from transients.

| $R_0$ ohms | $P$ mW | $T$ sec. | $N_h$ mW | $N$ mW | $F$ | $R_b$ ohms | $\tau$ sec. theory | $\tau$ sec. exper. |
|---|---|---|---|---|---|---|---|---|
| 500 | 13.0 | 11 | 0 | 13.0 | 1.80 | 2 500 | 5.0 | 5.2 |
| | | | 10 | 4.4 | 0.61 | 2 500 | 7.9 | 8.7 |
| | | | 10 | 4.4 | 0.61 | 100 | 18 | 15.6 |
| 50 | 35.6 | 10 | 20 | 18.4 | 1.37 | 250 | 5.3 | 5.6 |
| | | | 40 | 1.2 | 0.09 | 4 000 | 9.1 | 9.1 |

b) *Sinusoidal variations*

With the aid of the ultralow-frequency generator, described above, an experimental determination was made in different operating points: 1) of the thermistor impedance for small superimposed sinusoidal voltages on the thermistor side, 2) of the frequency dependence of

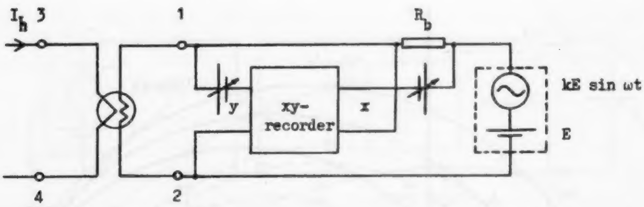



*Fig. 27.* Determination of the thermistor impedance for small superimposed sinusoidal voltages. Fig. 19 shows the principle of the ultralow-frequency generator.

a small variation of the thermistor voltage, caused by a small sinusoidal voltage superimposed on the heater voltage.

The output voltage of the ultralow-frequency generator was of the form $e = E + k\,E \sin \omega t$, where $E$ and $k$ could be varied independently of each other. At first, with $k = 0$, $E$ was chosen so that the desired operating point was obtained. Then a small A. C. voltage was superimposed by regulating $k$ to a suitable value, generally 5 to 10 percent.

The circuit of *Fig. 27* was employed for the impedance measurements. The $y$-amplifier of the $xy$-recorder was supplied with the thermistor voltage variations. The thermistor current variations caused variations of the voltage over $R_b$ and these variations were brought to the $x$-amplifier of the recorder. The two variable emf's connected in series with the input to the recorder compensate for the constant parts of voltage and current. For each frequency the relation between voltage and current was recorded. From the ellipse obtained the absolute value and the phase angle of $Z$ were easily calculated.

According to eq. (42) the thermistor impedance can be written

$$Z\,(\omega) = R_0 \frac{1 - F + j\,\omega\,T}{1 + F + j\,\omega\,T}$$

In the complex plane $Z$ is represented by a half circle (see Parts 1 and 2). For $\omega = 0$ we have $Z = R_0\,(1 - F)\,/\,(1 + F) = r$; for $\omega = \infty$ we have $Z = R_0$. In the range $F > 1$ the phase angle of $Z$ goes from $180°$ to $0°$ as $\omega$ goes from 0 to $\infty$.

*Fig. 28* gives theoretical and experimental impedance diagrams for a number of operating points, corresponding to one and the same value of the static resistance. As in Part 2, for the directly heated

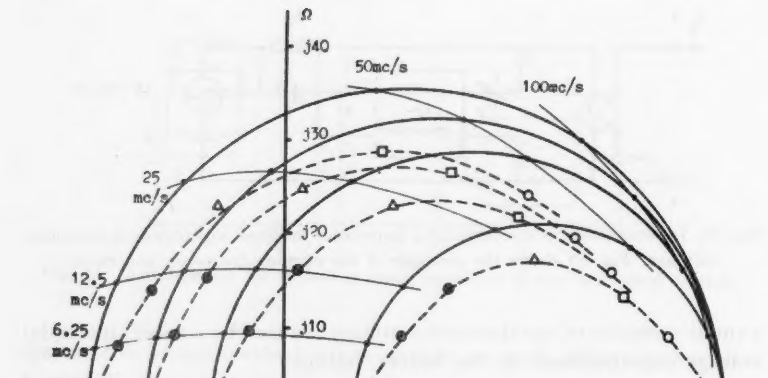



*Fig. 28.* Impedance diagram for the thermistor of Fig. 23. In all cases the static resistance of the thermistor is 50 ohms. 1 mc/s = 0.001 c/s.

a) $N_h = 0$ ; $N = 35.6$ mW
b) $N_h = 10$ mW; $N = 27.0$ mW
c) $N_h = 20$ mW; $N = 18.4$ mW
d) $N_h = 30$ mW; $N = 9.8$ mW

thermistor, we find that an experimentally determined curve always lies inside the corresponding theoretical curve. The experimental value of the phase angle is always smaller than the theoretical value. Mostly the absolute value of the impedance is also smaller than the theoretical value. At low frequencies it can be larger.

The arrangement of *Fig. 29* was used for recording the small thermistor voltage, caused by the superposition of a small sinusoidal voltage on the heater side. At each frequency two cycles of the superimposed thermistor voltage were recorded, and from the records the amplitude and phase characteristics were determined.

According to theory (cf. eq. (60)) the superimposed thermistor voltage should vary as $1/(1 + j\,\omega\tau)$. This means that: 1) at the critical frequency $f_0$, determined by $2\pi f_0 \tau = 1$, the amplitude has decreased 3 dB, the phase angle 45°, 2) at high frequencies the amplitude varies as $1/\omega\tau$, i. e., it decreases 6 dB per octave.

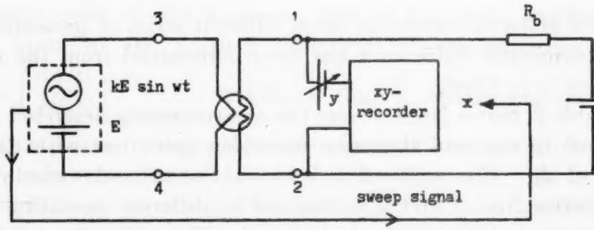



*Fig. 29.* **Arrangement for determining the amplitude and phase characteristics when superimposing a small A. C. voltage on the heater voltage.**

***Fig. 30*** **shows three pairs of amplitude and phase characteristics, corresponding to three cases with the same operating point but different values of the series resistance $R_b$. The curves are the theoretical characteristics, corresponding to the values of $f_0$ stated in the figure. The measured points are very close to the curves. At all measurements performed this was found to be the case.**

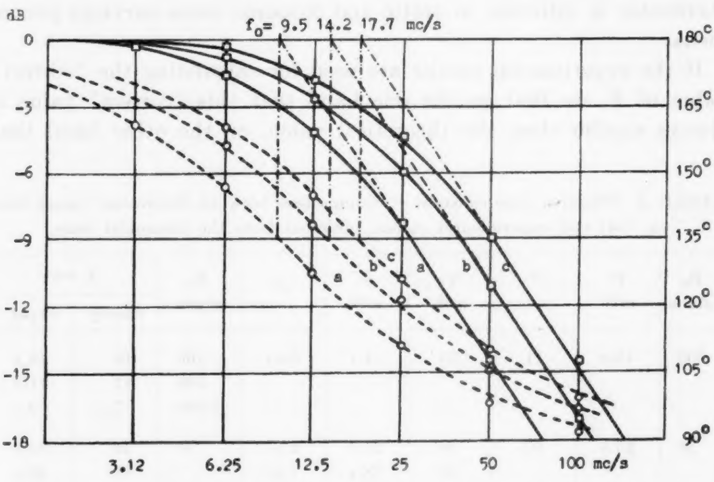



*Fig. 30.* **Amplitude and phase characteristics for the thermistor of Fig. 23. Same operating point in all cases.**

**1 mc/s = 0.001 c/s. $R_0$ = 500 ohms.**

**a) $R_b$ = 100 ohms; b) $R_b$ = 500 ohms; c) $R_b$ = 2 500 ohms.**

**——— = relative amplitude } of the small superimposed**
**– – – = phase angle } thermistor voltage**

*Table 2* gives the results of three different series of measurements. The experimental value of $\tau$ has been determined from the critical frequency, $\tau = 1/2\pi f_0$.

In Table 2, Series No. 1 covers the measurements described above, performed in one and the same operating point but with different values of $R_b$. The agreement between theory and experiment is good. Series Nos. 2 and 3 correspond to different operating points with the same static resistance but with different heater powers. In Series No. 2 the series resistance $R_b$ is equal to the static resistance, in Series No. 3 the series resistance is much larger. For $R_b = R_0$ the agreement is very good in all cases, for $R_b \gg R_0$ there are appreciable deviations in those cases where the thermistor power is large.

A closer study of the experimental results of Tables 1 and 2, and of the experimental impedance diagrams, shows that the deviations between theory and experiment can be explained by the hypothesis that the value of the dynamic factor $F$ is different under static and dynamic conditions. In other words: the dynamic resistance of the thermistor is different in static and dynamic (time-varying) phenomena.

If the experimental results are used for calculating the "correct" value of $F$, we find on the one hand that this "correct" value is always smaller than the theoretical value, on the other hand that

TABLE 2. Effective time constant $\tau$. Comparison between theoretical values from eq. (54) and experimental values, obtained from the sinusoidal state.

| $R_0$ ohms | $P$ mW | $T$ sec. | $N_h$ mW | $N$ mW | $F$ | $R_b$ ohms | $\tau$ sec. theory | $\tau$ sec. exper. |
|---|---|---|---|---|---|---|---|---|
| 500 | 13.0 | 11 | 10 | 4.4 | 0.61 | 100 | 18 | 16,8 |
| | | | | | | 500 | 11 | 11.2 |
| | | | | | | 2 500 | 7.9 | 9.0 |
| 50 | 35.6 | 10 | 10 | 27.0 | 2.01 | 50 | 10 | 10.0 |
| | | | 20 | 18.4 | 1.37 | | 10 | 10.1 |
| | | | 30 | 9.8 | 0.73 | | 10 | 10.0 |
| | | | 40 | 1.2 | 0.09 | | 10 | 10.1 |
| 50 | 35.6 | 10 | 10 | 27.0 | 2.01 | 4 000 | 3.4 | 4.5 |
| | | | 20 | 18.4 | 1.37 | | 4.3 | 5.2 |
| | | | 30 | 9.8 | 0.73 | | 5.9 | 6.8 |
| | | | 40 | 1.2 | 0.09 | | 9.1 | 9.2 |

it depends both on the frequency and on the series resistance $R_b$. If we turn to the original definition of the dynamic factor, $F = -\left(\frac{dR/R}{dN/N}\right)_{N_h = \text{const.}}$, it follows that a certain change in the thermistor power during a dynamic course of events always gives a smaller resistance change than expected. Of course, this is due to the fact that we have neglected that under dynamic conditions the temperature distribution is uneven.

It is evident that this uneven temperature distribution must depend both on the frequency and on the properties of the circuit in which the thermistor is inserted. The measurements agree well with theory in those cases where the product $F \frac{R_b - R_0}{R_b + R_0}$ is $\ll 1$, i. e., where $R_b$ is about equal to $R_0$ or where the thermistor power $N$ is so small that $F \approx 0$.

# Bibliography

1. BOLLMAN, J. H.: A pilot-channel regulator for the K1 carrier system. Bell Lab. Record *20* (1942) 258.

2. BROOKS, F. A.: The K2 carrier system. Bell Lab. Record *27* (1949) 141.

3. SCHRADER, H.-J.: Wechselstrom-Kompensationsmessungen mit indirekt geheizten NTC-Widerständen. E. T. Z., Ausg. A. *73* (1952) 547.

4. EKELÖF, S. and KIHLBERG, G.: Theory of the thermistor as an electric circuit element (A study of thermistor circuits — 1). Trans. Chalmers Univ. of Technology, No. 142 (1954) (also: Acta Polyt. No. 154 (1954)).

5. BJÖRK, N. and DAVIDSON, R.: Small signal behaviour of directly heated thermistors (A study of thermistor circuits — 2). Trans. Chalmers Univ. of Technology, No. 169 (1955) (also: Acta Polyt. No. 186 (1955)).

6. WIDDIS, F. C.: The indirectly heated thermistor as a precise A. C.—D. C. transfer device. Proc. Instn Elect. Engrs, Part B *103* (1956) 693.

7. EKELÖF, S., BJÖRK, N. and DAVIDSON, R.: Large signal behaviour of directly heated thermistors (A study of thermistor circuits — 3). Trans. Chalmers Univ. of Technology, No. 185 (1957) (also: Acta Polyt. No. 216 (1957)).

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